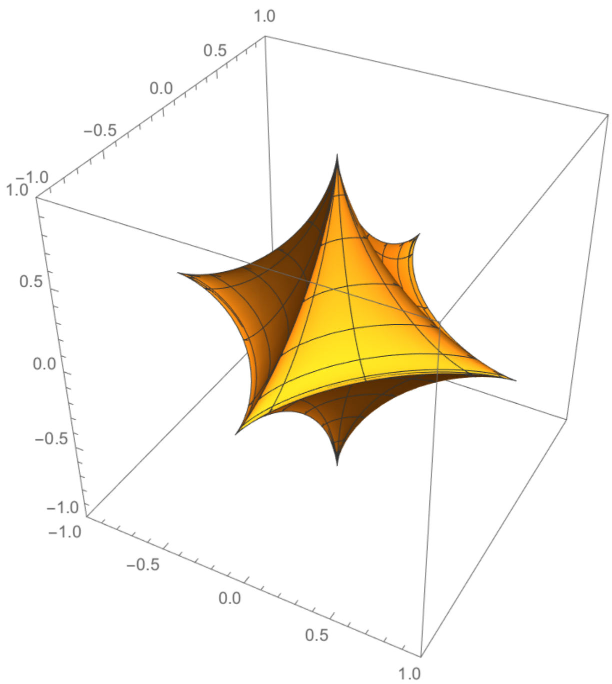

Gaussian Curvature

An Interesting Property Of Quadratic Polynomial

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Contents

0.1 Acknowledgements

Another article from Team GC.This article is on "An Interesting Property Of Quadratic Polynomial" is intended for inequalities and it has been authored by Do Xuan Trong and some modification & edition by Aritra12 We are also thankful to the several users on AoPS who posted problems and solution. No handout can be all perfect so if you see any problem or typos don't forget to gmail at gaussiancurvature360@gmail.com

Chapter 1

An Interesting Property Of Quadratic Polynomial

1.1 Lemmas

(Lemma 1)

Consider quadratic polynomial $f(x) = x^2 + bx + c$ where b, c are real numbers. Then $f(x) \geq 0$ holds for all $x \geq 0$ if and only if $b^2 \leq 4c$, or $b^2 > 4c$ and $b,\ c \geq 0$.

Proof. If $b^2 \leq 4c$ then the inequality is trivial. If $b^2 > 4c$, then $f(x)$ must have two real roots $x_1 < x_2$ such that

$$x_1 + x_2 = -b,\ x_1x_2 = c.$$

So $f(x) \geq 0,\ \forall x \geq 0$ if and only if $x_1 < x_2 \leq 0$ and then $b,\ c \geq 0$, as desired.

(Lemma 2)

Consider quadratic polynomial $f(x) = ax^2 + bx + c$ where a, b, c are real numbers and $a > 0$. Then $f(x) \geq 0$ holds for all $x \geq 0$ if and only if $b^2 \leq 4ac$, or $b^2 > 4ac$ and $b,\ c \geq 0$.

Proof. It's corollary of lemma 1 if we replace $(b, c) \to \left(\frac{b}{a}, \frac{c}{a}\right)$.

(Lemma 3)

Consider quadratic polynomial $f(x) = ax^2 - bx + c$ where a, b, c are real numbers and $a > 0$. Let $p \leq q$ be real numbers, then $f(x) \geq 0$ holds for all $x \in [p; q]$ if and only if $b^2 \leq 4ac$, or

$$\begin{cases} b^2 > 4ac, \\ ap^2 - bp + c \geq 0, \\ b \leq 2ap \end{cases} \quad \text{or} \quad \begin{cases} b^2 > 4ac, \\ aq^2 - bq + c \geq 0, \\ b \geq 2aq \end{cases}$$

1.2 Examples

Example 1.2.1 (Vasile Cirtoaje). Let a,b,c be real numbers such that $abc \geq 0$. Prove that

$$a^2+b^2+c^2+2abc+4 \geq 2(a+b+c)+ab+bc+ca.$$

Proof. We can assume that $(b-1)(c-1) \geq 0$. The inequality is

$$f(a) = a^2+(2bc-b-c-2)+b^2-bc+c^2-2(b+c)+4 \geq 0.$$

We have

$$\begin{aligned}\Delta_a &= (2bc-b-c-2)^2-4\left[b^2-bc+c^2-2(b+c)+4\right]\\ &= 4(bc-3)(b-1)(c-1)-3(b-c)^2.\end{aligned}$$

If $\Delta_a \leq 0$ then the inequality is proved. If $\Delta_a > 0$ then $bc > 3$ and $a \geq 0$ as $abc \geq 0$. Applying the first lemma, we need to prove

$$2bc-b-c-2 \geq 0, \quad b^2-bc+c^2-2(b+c)+4 \geq 0.$$

Indeed

$$b^2-bc+c^2-2(b+c)+4 \geq \frac{(b+c)^2}{4}-2(b+c)+4 = \left(\frac{b+c}{2}-2\right)^2 \geq 0.$$

Assume $2bc-b-c-2 < 0$, then since $\Delta_a > 0$ we get

$$b+c+2-2bc > 2\sqrt{b^2-bc+c^2-2(b+c)+4} \geq 2\left(\frac{b+c}{2}-2\right) \Longrightarrow 2bc < 6 \Longleftrightarrow bc < 3,$$

which contradicts. Hence $2bc-b-c-2 \geq 0$, as desired.

Example 1.2.2 (Vasile Cirtoaje). Let $a,b,c \geq 0$ and $0 \leq k \leq \sqrt{2}$. Prove that

$$a^2+b^2+c^2+kabc+2k+3 \geq (k+2)(a+b+c).$$

Proof. The inequality is linear function of k. So it's enough to prove when $k \in \left\{0;\sqrt{2}\right\}$. If $k=0$, it becomes

$$(a-1)^2+(b-1)^2+(c-1)^2 \geq 0.$$

Let's see the case $k=\sqrt{2}$. It can be written as

$$f(a) = a^2+\left(bc\sqrt{2}-2-\sqrt{2}\right)+b^2+c^2+2\sqrt{2}+3-\left(2+\sqrt{2}\right)(b+c) \geq 0.$$

Suppose $(b-1)(c-1) \geq 0$ and consider the case $\Delta_a > 0$ (for example, $b=c=2$), which is

$$\left(bc\sqrt{2}-2-\sqrt{2}\right)^2 > 4\left[b^2+c^2+2\sqrt{2}+3-\left(2+\sqrt{2}\right)(b+c)\right].$$

We have

$$\begin{aligned} b^2+c^2+2\sqrt{2}+3-\left(2+\sqrt{2}\right)(b+c) &\geq \frac{(b+c)^2}{2}+2\sqrt{2}+3-\left(2+\sqrt{2}\right)(b+c) \\ &= \left(\frac{b+c}{\sqrt{2}}-\sqrt{2}-1\right)^2 \geq 0. \end{aligned}$$

and the rest is $bc\sqrt{2}-2-\sqrt{2}\geq 0$. Assume $bc\sqrt{2}-2-\sqrt{2}<0$, then

$$\begin{aligned} 2+\sqrt{2}-bc\sqrt{2} > 2\sqrt{b^2+c^2+2\sqrt{2}+3-\left(2+\sqrt{2}\right)(b+c)} &\geq 2\sqrt{\left(\frac{b+c}{\sqrt{2}}-\sqrt{2}-1\right)^2} \\ &\geq 2\left(\sqrt{2}+1-\frac{b+c}{\sqrt{2}}\right). \end{aligned}$$

This leads to

$$\sqrt{2}(b+c)\geq\sqrt{2}bc+\sqrt{2}\Leftrightarrow\sqrt{2}(b-1)(c-1)<0,$$

which is a contradiction. Hence $bc\sqrt{2}-2-\sqrt{2}\geq 0$, as desired.

Example 1.2.3 (Tran Nam Dung). Find the smallest real number k such that

$$abc+2+k\left[(a-1)^2+(b-1)^2+(c-1)^2\right]\geq a+b+c$$

holds for all $a,b,c\geq 0$.

Proof [Hint]. Choose $c=0$ and $a=b=1+\frac{1}{\sqrt{2}}$ to get $k\geq\frac{1}{\sqrt{2}}$.

Example 1.2.4. Let $a,b,c>0$. Prove that

$$(a+b+c-3)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-3\right)+abc+\frac{1}{abc}\geq 2.$$

Proof. Multiply both sides by abc and write it as

$$f(a)=\left(b^2c^2+b+c-3bc\right)a^2+\left[b^2+10bc+c^2-3(bc+1)(b+c)\right]a+bc(b+c)-3bc+1\geq 0.$$

By the *AM - GM* inequality

$$b^2c^2+b+c-3bc\geq b^2c^2+2\sqrt{bc}-3bc=\sqrt{bc}\left(\sqrt{bc}-1\right)^2\left(\sqrt{bc}+2\right)\geq 0,$$

$$bc(b+c)-3bc+1\geq 2bc\sqrt{bc}-3bc+1=\left(\sqrt{bc}-1\right)^2\left(2\sqrt{bc}+1\right)\geq 0.$$

Notice that $b^2c^2+b+c-3bc=0 \Leftrightarrow b=c=1$ and then inequality becomes equality. Otherwise, we calculate

$$\begin{aligned}\Delta_a &= \left[b^2+10bc+c^2-3(bc+1)(b+c)\right]^2-4\left(b^2c^2+b+c-3bc\right)\left[bc(b+c)-3bc+1\right]\\ &=(b-1)^2(c-1)^2\left[b^2+14bc+c^2-4(bc+1)(b+c)\right].\end{aligned}$$

In the case $\Delta_a>0$ we get

$$b^2+14bc+c^2>4(bc+1)(b+c),$$

and because $(bc+1)(b+c)\geq 4bc$ (it's just *AM - GM*), we obtain

$$b^2+10bc+c^2>3(bc+1)(b+c),$$

this is what we want.

1.3 The pqr method

This method has become quite popular. The idea is that when solving three-variable inequalities, we set $p=a+b+c$, $q=ab+bc+ca$ and $r=abc$. The following result is important.

$$\frac{p\left(9q-2p^2\right)-2\sqrt{\left(p^2-3q\right)^3}}{27}\le r\le\frac{p\left(9q-2p^2\right)+2\sqrt{\left(p^2-3q\right)^3}}{27}.$$

This follows from

$$(a-b)^2(b-c)^2(c-a)^2=p^2q^2-4q^3+\left(18pq-4p^3\right)r-27r^2\ge 0$$

Let's see its application in the following problem.

Example 1.3.1 (Do Xuan Trong). Let a,b,c be real numbers and no two of which are equal. Prove that

$$\frac{1}{(a-b)^2}+\frac{1}{(b-c)^2}+\frac{1}{(c-a)^2}\ge\frac{4(ab+bc+ca)}{a^2b^2+b^2c^2+c^2a^2}.$$

Proof. If $ab+bc+ca\le 0$ then the inequality is clear. If $ab+bc+ca>0$, let $p=a+b+c=1$, $q=ab+bc+ca$ and $r=abc$. We have $a^2b^2+b^2c^2+c^2a^2=q^2-2r$ and

$$\frac{1}{(a-b)^2}+\frac{1}{(b-c)^2}+\frac{1}{(c-a)^2}=\left(\frac{1}{a-b}+\frac{1}{b-c}+\frac{1}{c-a}\right)^2=\frac{\left(a^2+b^2+c^2-ab-bc-ca\right)^2}{(a-b)^2(b-c)^2(c-a)^2}$$
$$=\frac{(1-3q)^2}{q^2-4q^3+(18q-4)r-27r^2}.$$

The inequality is

$$\frac{(1-3q)^2}{q^2-4q^3+(18q-4)r-27r^2}\ge\frac{4q}{q^2-2r},$$

which is

$$f(r)=108qr^2-2(9q-1)(5q-1)r+q^2(5q-1)^2\ge 0.$$

Since $108q>0$, we calculte

$$\Delta'_r=(9q-1)^2(5q-1)^2-108q^3(5q-1)^2$$
$$=(5q-1)^2(3q-1)^2(1-12q).$$

If $q\ge\frac{1}{12}$, we have Q.E.D. If $q<\frac{1}{12}$, since

$$\frac{9q-2-2\sqrt{(1-3q)^3}}{27}=r_1\le r\le r_2=\frac{9q-2+2\sqrt{(1-3q)^3}}{27},$$

and the lemma 3, we will show

$$\begin{cases}f(r_2)\ge 0,\\ 2(9q-1)(5q-1)\ge 2\cdot 108q\cdot r_2\end{cases}\Leftrightarrow\begin{cases}f(r_2)\ge 0,\\ (9q-1)(5q-1)\ge 4q\left[9q-2+2\sqrt{(1-3q)^3}\right]\end{cases}$$

The condition $f(r_2) \geq 0$ is equivalent to prove $f(r) \geq 0$ when two numbers are equal, and this is true. The second condition is equivalent to

$$q < \frac{\sqrt{265}-3}{128},$$

this is true because $\frac{\sqrt{265}-3}{128} > \frac{1}{12}$.

Example 1.3.2. Let x, y and z be positive numbers such that $x^3+y^3+z^3+xyz=4$ Prove that:

$$\frac{x^2+y^2}{x+y}+\frac{y^2+z^2}{y+z}+\frac{z^2+x^2}{z+x} \geq 3$$

Proof. Write inequality as

$$\frac{x^2+y^2}{x+y}+\frac{y^2+z^2}{y+z}+\frac{z^2+x^2}{z+x} \geqslant 3\sqrt[3]{\frac{x^3+y^3+z^3+xyz}{4}}$$

Suppose $p=x+y+z=3, q=xy+yz+zx=3-3t^2(0 \leqslant t < 1)$ and $r=abc$ inequality become

$$\frac{2p^2q-4pr-2q^2}{pq-r} \geqslant 3\sqrt[3]{\frac{p^3-3pq+4r}{4}}$$

equivalent to

$$\frac{36-12r-18t^2\left(t^2+1\right)}{9\left(1-t^2\right)-r} \geqslant 3\sqrt[3]{r+\frac{27}{4}t^2}$$

Or

$$4-\frac{6\left(1-t^2\right)\left(4-t^2\right)}{9\left(1-t^2\right)-r} \geqslant \sqrt[3]{r+\frac{27}{4}t^2}$$

Because $r \leqslant (1+2t)(1-t)^2$, so

$$4-\frac{6\left(1-t^2\right)\left(4-t^2\right)}{9\left(1-t^2\right)-r} \geqslant 4-\frac{6\left(1-t^2\right)\left(4-t^2\right)}{9\left(1-t^2\right)-(1+2t)(1-t)^2} = \frac{3t^2+t+2}{t+2}$$

and

$$r+\frac{27}{4}t^2 \leqslant (1+2t)(1-t)^2+\frac{27}{4}t^2 = 2t^3+\frac{15}{4}t^2+1$$

Therefore we need to show that

$$\frac{3t^2+t+2}{t+2} \geqslant \sqrt[3]{2t^3+\frac{15}{4}t^2+1}$$

Which is true because

$$\left(\frac{3t^2+t+2}{t+2}\right)^3-\left(2t^3+\frac{15}{4}t^2+1\right) = \frac{\left(4t^2+5t+6\right)(5t-2)^2t^2}{4(t+2)^3} \geqslant 0$$

Example 1.3.3 (Do Xuan Trong). Let $a,b,c \geq 0$. Prove that

$$2\sqrt{2}(a-b)(b-c)(c-a) \leq \frac{a^4+b^4+c^4}{a+b+c} - \frac{3abc\left(a^2+b^2+c^2\right)}{(a+b+c)^2}.$$

Proof. Let $p=a+b+c=1$, $q=ab+bc+ca$ and $r=abc$. Because the right hand side is non-negative, we consider the case $a \leq b \leq c$. Hence

$$(a-b)(b-c)(c-a) = \sqrt{q^2-4q^3+(18q-4)r-27r^2}.$$

We have

$$\frac{a^4+b^4+c^4}{a+b+c} - \frac{3abc\left(a^2+b^2+c^2\right)}{(a+b+c)^2} = 2q^2-4q+1+(6q+1)r.$$

So the inequality becomes

$$\left(2q^2-4q+1+(6q+1)r\right)^2 \geq 8\left[q^2-4q^3+(18q-4)r-27r^2\right],$$

$$\Leftrightarrow f(r) = \left(36q^2+12q+217\right)r^2+2\left(12q^3-22q^2-70q+17\right)r+\left(2q^2+4q-1\right)^2 \geq 0.$$

We see

$$\begin{aligned}\Delta'_r &= \left(12q^3-22q^2-70q+17\right)^2-\left(36q^2+12q+217\right)\left(2q^2+4q-1\right)^2 \\ &= -8(3q-1)^2\left(16q^3+48q^2+28q-9\right)\end{aligned}$$

In the cases $16q^3+48q^2+28q-9<0$ we have (using calculator) $q<0.226683$. But then $12q^3-22q^2-70q+17>0$ (because this is $q<0.22849$ (note that $0<q<3$)). So the inequality is proved.

This result is equivalent to

$$a^4+b^4+c^4-\frac{3\left(a^2+b^2+c^2\right)abc}{a+b+c} \geq 2\sqrt{2}\left|a^3b+b^3c+c^3a-ab^3-bc^3-ca^3\right|.$$

This is stronger than the problem made by Pham Kim Hung

$$a^4+b^4+c^4-abc(a+b+c) \geq 2\sqrt{2}\left|a^3b+b^3c+c^3a-ab^3-bc^3-ca^3\right|.$$

We can use that result to prove (also created by me)

$$\frac{a^4+b^4}{a+b}+\frac{b^4+c^4}{b+c}+\frac{c^4+a^4}{c+a} \geq a^3+b^3+c^3+2\sqrt{2}(a-b)(b-c)(c-a).$$

1.4 Problems

Problem 1.4.1 (Do Xuan Trong). Let $a,b,c \geq 0$. Prove that

$$\frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a} \geq a+b+c+\frac{4\left|(a-b)(b-c)(c-a)\right|}{a^2+b^2+c^2}.$$

Hint. Make a lemma the same as example 7.

Problem 1.4.2. Let $a,b,c > 0$. Prove that

$$(a+b+c-3)(ab+bc+ca-3) \geq 3(abc-1)(a+b+c-ab-bc-ca).$$

Hint. Check the cases $c=0$ and $b=c$.

Problem 1.4.3 (Do Xuan Trong). Let $a,b,c \geq 0$. Prove that

$$14abc+3\left(a^2b^2+b^2c^2+c^2a^2\right)-8(ab+bc+ca)+2(a+b+c)+1 \geq 0.$$

Hint. Check the cases $c=0$ and $b=c$.